SPINORS AND SPACE-TIME

Volume 2

Spinor and twistor methods in
space–time geometry

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Twistors

6.1 The twistor equation and its solution space

At various places in Volume 1 we stressed the fact that the two-component spinor calculus is a very specific calculus for studying the structure of space–time manifolds. Indeed, the four-dimensionality and $(+ - - -)$ signature of space–time, together with the desirable global properties of orientability, time-orientability, and existence of spin structure, may all, in a sense, be regarded as *derived* from two-component spinors, rather than just given. However, at this stage there is still only a limited sense in which these properties can be so regarded, because the manifold of space–time points itself has to be given beforehand, even though the nature of this manifold is somewhat restricted by its having to admit the appropriate kind of spinor structure. If we were to attempt to take totally seriously the philosophy that all the space–time concepts are to be derived from more primitive spinorial ones, then we would have to find some way in which the space–time points themselves can be regarded as derived objects.

Spinor algebra by itself is not rich enough to achieve this, but a certain extension of spinor algebra, namely twistor algebra, can indeed be taken as more primitive than space–time itself. Moreover, it is possible to use twistors to build up other physical concepts directly, without the need to pass through the intermediary of space–time points. The programme of twistor theory, in fact, is to reformulate the whole of basic physics in twistor terms. The concepts of space–time points and curvature, of energy–momentum, angular momentum, quantization, the structure of elementary particles with their various internal quantum numbers, wave functions, space–time fields (incorporating their possibly nonlinear interactions), can all be formulated – with varying degrees of speculativeness, completeness, and success – in a more or less direct way from primitive twistor concepts.

Twistor theory has, however, become rather mathematically elaborate. To cover in any thorough and comprehensible way all the above-mentioned aspects of the theory would in itself require a book considerably larger than the present volume. (Some of these topics are to be covered in a forthcoming book by Ward and Wells, 1986.) In any case, to appreciate twistors fully and to be able to calculate with them, one must first study

spinor theory, somewhat along the lines that we have followed in Volume 1. Thus we do not attempt to be in any way complete in our description of twistor theory. The account given here will serve as an extended (perhaps somewhat lop-sided) introduction to the subject. We shall develop in detail mainly that part of twistor theory which relates to spinorial descriptions of twistors, and indicate some of their profound connections with energy-momentum/angular momentum and with massless fields. We do not enter into much discussion of how twistors may be regarded as more primitive than space–time points, nor do we discuss quantization at any length, or particle theory, or give much detail of the treatment of nonlinear fields. In this chapter, apart from giving a discussion of local twistors in general curved space–time (local twistors in fact lying somewhat outside the main development of twistor theory), we shall restrict our account of twistors almost entirely to Minkowski space–time $\mathbb{M}$, though many interrelations with curved space–time properties will be given. In §7.4 we indicate how certain twistor ideas can be applied in general space–time $\mathcal{M}$ (particularly in relation to a hypersurface $\mathcal{H} \subset \mathcal{M}$), and in §9.5 we show how twistors can be used in cosmological models, while in §9.9 we introduce the concept of 2-*surface twistors* and show how our flat-space discussion of §§6.3–6.5 can be adapted to a curved-space context and suggest a quasi-local (and asymptotic) definition of mass–momentum–angular momentum surrounded by a 2-surface (the asymptotic mass–momentum agreeing with the standard definition of Bondi–Sachs). However we have had to omit a good deal of the detailed theory of *general* curved-space twistor theory. The twistorial description of space–time curvature is one of the more elaborate and sophisticated (and, indeed, remarkable – though incomplete) parts of twistor theory and, regrettably, it must remain outside the scope of the present work. (See Penrose 1976a, 1979a, Hansen, Newman, Penrose and Tod 1978, Tod 1980, Tod and Ward 1979, Ward 1978, Penrose and Ward 1980, Porter 1982, Hitchin 1979, Atiyah, Hitchin and Singer 1978, Wells 1982 for details.) There is much that is striking and illuminating even in the application of twistor theory to the weak-field limit of general relativity, *cf.* §§6.4, 6.5. This will be of crucial significance also for the curved-space discussion of §9.9. (See also Huggett and Tod 1985).

The twistor equation

Our point of departure is the equation (*cf.* (4.12.46), (5.6.38)*)

$$\nabla_{A'}^{(A}\omega^{B)} = 0, \tag{6.1.1}$$

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called the *twistor equation* (Penrose 1967a; *cf.* also Gårding 1945)*. We begin by investigating its formal properties, leaving its physical and geometrical significance to later sections. First, we easily prove it is conformally invariant. Choosing

$$\hat{\omega}^B = \omega^B, \qquad (6.1.2)$$

we get, from (5.6.15), (5.6.14)

$$\hat{\nabla}_{AA'}\hat{\omega}^B = \nabla_{AA'}\omega^B + \varepsilon_A{}^B \Upsilon_{CA'}\omega^C, \qquad (6.1.3)$$

whence

$$\hat{\nabla}_{A'}^{(A}\hat{\omega}^{B)} = \Omega^{-1}\nabla_{A'}^{(A}\omega^{B)}. \qquad (6.1.4)$$

Thus, conformal invariance is established.

There is a severe consistency condition for (6.1.1) in curved space–time, analogous to (5.8.2). For we have (*cf.* (4.9.2), (4.9.11), (4.6.35), (5.1.44))

$$\nabla^{A'(C}\nabla_{A'}^A\omega^{B)} = -\Box^{(CA}\omega^{B)} = -\Psi^{CA}{}_D{}^B\omega^D - ie\varphi^{(CA}\omega^{B)}, \qquad (6.1.5)$$

allowing for the presence of an electromagnetic field, and the possibility that ω^B has charge e. Thus (6.1.1) implies

$$\Psi_{ABCD}\omega^D = -ie\varphi_{(AB}\omega_{C)}, \qquad (6.1.6)$$

which is the analogue of (5.8.2). If $\omega^D \neq 0$ and $e = 0$, we see, by reference to Proposition (3.5.26), that ω^D is a four-fold principal spinor of Ψ_{ABCD}. Thus, non-zero uncharged solutions of the twistor equation can exist only at points where Ψ_{ABCD} is either zero or 'null' (i.e. possessing a four-fold principal spinor). If $e \neq 0$, the situation is no better. In view of these difficulties our discussion of (6.1.1) in this chapter will be restricted to conformally flat space–times (characterized by $\Psi_{ABCD} = 0$; see §§6.8,9), and most of our calculations will actually be done in Minkowski space $\mathbb{M}$. Their extension to conformally flat space then follows from conformal invariance. (For extensions to arbitrary curved space: local twistors, hypersurface twistors and 2-surface twistors, see §6.9, §7.4 and §9.9, respectively.) Even in flat space, (6.1.6) has no solutions other than zero if $e \neq 0$ and φ_{AB} is somewhere non-vanishing. *So we assume, from now on, that unless the contrary is stated all fields are uncharged.*

In Minkowski space, equation (6.1.1) indeed possesses non-trivial solutions. We shall now find these explicitly. We choose an arbitrary origin O in $\mathbb{M}$ and label points by their position vectors x^a relative to O. We regard x^a as a vector field on $\mathbb{M}$. At O it is zero, and everywhere it satisfies

$$\nabla_a x^b = g_a{}^b, \qquad (6.1.7)$$

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* A version of this equation, written in terms of γ-matrices, was found by Wess and Zumino (1974) in connection with supersymmetry theory. See Appendix to this volume: (B.94), (B.95).

since in standard Minkowski coordintes, x^a, the components of x^a at a point, are the coordinates of that point. Now consider

$$\nabla_{A'}^A \nabla_{B'}^B \omega^C, \tag{6.1.8}$$

ω^C being a solution of (6.1.1). The expression is therefore skew in BC. But since $\mathbb{M}$ is flat, we can commute the derivatives and then the expression is seen to be skew in AC. It is therefore totally skew in ABC and so must vanish. This tells us that $\nabla_{B'}^B \omega^C$ is constant. Since it is skew, it must therefore be a constant multiple of ε^{BC}, say $-i\pi_{B'}\varepsilon^{BC}$ for some constant spinor $\pi_{B'}$. (The factor $-i$ is inserted for later convenience.) So we have

$$\nabla_{BA'}\omega^C = -i\varepsilon_B{}^C \pi_{A'}. \tag{6.1.9}$$

Integrating this equation gives $\omega^C = x^{BA'}(-i\varepsilon_B{}^C \pi_{A'}) + $ constant, as can be seen by writing it in coordinate form, and so we find

$$\left.\begin{array}{c} \omega^A = \mathring{\omega}^A - ix^{AA'}\mathring{\pi}_{A'} \\ \pi_{A'} = \mathring{\pi}_{A'} \end{array}\right\} \tag{6.1.10}$$

where $\mathring{\omega}^A$ and $\mathring{\pi}_{A'}$ are to be understood as follows: since ω^A is a spinor *field*, the RHS of (6.1.10) (1) must be regarded as a spinor field also. This can be done by regarding $\mathring{\omega}^A$ and $\mathring{\pi}_{A'}$ as *constant spinor fields* whose values coincide with those of ω^A and $\pi_{A'}$, respectively, at the origin. A similar convention should be understood whenever we write a point symbol over a spinor kernel. (The symbol '∘' above $\pi_{A'}$ is, of course, redundant here, but it makes what follows more consistent.)

Twistor space

As in the case of all (complex) linear differential equations, the solutions of the twistor equation constitute a vector space over the complex numbers, with scalar multiplication and addition of solutions defined in the obvious way. In the case of a general linear equation this vector space is often infinite-dimensional. It is clear from (6.1.10), however, that the solutions ω^A of the twistor equation are fully determined by the four complex components at O of ω^A and $\pi_{A'}$ in a spin-frame at O. These solutions ω^A therefore constitute a *four-dimensional* vector space $\mathbb{T}^\alpha$ over the complex numbers called *twistor space* (and they thus have eight real degrees of freedom). The elements of twistor space are called $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$-twistors, and we shall usually denote them by sans-serif capital kernel symbols with small Greek (four-dimensional) abstract indices, e.g., Z^α. If we denote the particular solution ω^A of (6.1.1) by Z^α we express this as follows:

$$\mathsf{Z}^\alpha = [\omega^A]. \tag{6.1.11}$$

Multiplication by a complex number and addition of twistors are defined in the obvious way:

$$\lambda[\omega^A] = [\lambda\omega^A] \quad (\lambda \in \mathbb{C}), \quad [\omega^A] + [\xi^A] = [\omega^A + \xi^A]. \quad (6.1.12)$$

From these $[^1_0]$-twistors we can build up twistors of arbitrary valence $[^p_q]$ according to the standard rules of constructing tensor systems such as given in Chapter 2.

Thus we have abstract-index copies $\mathbb{T}^\beta, \mathbb{T}^\gamma, \ldots,$ of $\mathbb{T}^\alpha$, and other spaces $\mathbb{T}_\alpha$, $\mathbb{T}^{\gamma\delta}_\beta, \ldots$ It turns out, however, that higher-valence twistors cannot in general be represented by single fields of spinors. In order to make the algebra of higher-valence twistors more systematic and manageable in terms of their spinor-field descriptions, it is much more convenient to use the *pair* of spinor fields ω^A, $\pi_{A'}$ to represent $\mathbb{Z}^\alpha$ rather than to use ω^A alone. When concerned with such descriptions we shall, as an alternative to (6.1.11), write*

$$\mathbb{Z}^\alpha = (\omega^A, \pi_{A'}), \quad (6.1.13)$$

where ω^A and $\pi_{A'}$ are related by (6.1.9) (or, equivalently, by (6.1.10)). But, unlike (6.1.11), the description (6.1.13) is not conformally invariant. (However, see §6.9.)

Since knowing ω^A is fully equivalent to knowing the constant spinors $\mathring{\omega}^A$ and $\mathring{\pi}_{A'}$ (*cf.* (6.1.9), (6.1.10)), we can also represent the field ω^A and hence $\mathbb{Z}^\alpha$ by the *values* $\omega^A(O)$ and $\pi_{A'}(O)$ of the spinor fields ω^A and $\pi_{A'}$ at O. We then write

$$\mathbb{Z}^\alpha \overset{O}{\leftrightarrow} (\omega^A(O), \pi_{A'}(O)), \quad (6.1.14)$$

the symbol $\overset{O}{\leftrightarrow}$ reminding us that the correspondence (6.1.14) is not Poincaré-invariant, but depends on the choice of a particular space–time origin O. Occasionally we shall use the notation

$$\mathbb{Z}^A = \omega^A, \quad \mathbb{Z}_{A'} = \pi_{A'}, \quad (6.1.15)$$

where ω^A, $\pi_{A'}$ could be either spinor fields (description (6.1.13)) or point-spinors at O (description (6.1.14)). They are, in either case, called the 'spinor parts' of $\mathbb{Z}^\alpha$ ('at O'). By (6.1.14) and (6.1.10) we have

$$\lambda(\omega^A, \pi_{A'}) = (\lambda\omega^A, \lambda\pi_{A'}), \quad (\omega^A, \pi_{A'}) + (\xi^A, \eta_{A'}) = (\omega^A + \xi^A, \pi_{A'} + \eta_{A'}).$$
$$(6.1.16)$$

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* Any temptation to identify the twistor (6.1.13) with a Dirac spinor (*cf.* (5.10.15) and the Appendix) should be resisted. Though there is a certain formal resemblance *at one point*, the vital twistor dependence (6.1.10) on position has no place in the Dirac formalism.

One might choose to regard $\mathbb{T}^\alpha$ (non-Poincaré-invariantly) as the direct sum $\mathfrak{S}^A[O] \oplus \mathfrak{S}_{A'}[O]$ of the spaces of spinors of type ω^A and $\pi_{A'}$ at O. (We recall that $\mathfrak{S}^{\mathscr{A}}[P]$ is the complex vector space of spinors of index type $\mathscr{A}$ at the point P.) The index α of Z^α would then be viewed as a kind of direct sum of the abstract index A and the abstract index A' in reverse position. (Note that this is quite different from the 'clumping' of §2.2.) In practice we can often treat A and A' as the two 'values' taken by α. When viewed in this way, α is somewhat intermediate between being fully abstract and being numerical. Note that the 'components' (6.1.15) corresponding to these 'values' A and A' of α get transformed when we change the origin (*cf.* (6.1.10)), so we prefer not to adopt this view formally.

If we choose an arbitrary spin-frame (o^A, ι^A) at O, we may construct a twistor basis as follows:

$$\delta_0^\alpha \overset{o}{\leftrightarrow} (o^A, O) \qquad \delta_1^\alpha \overset{o}{\leftrightarrow} (\iota^A, O)$$

$$\delta_2^\alpha \overset{o}{\leftrightarrow} (O, -\iota_{A'}) \qquad \delta_3^\alpha \overset{o}{\leftrightarrow} (O, o_{A'}). \qquad (6.1.17)$$

The linear independence of these twistors is manifest. Now, since

$$(\omega^A(O), \pi_{A'}(O)) = (\omega^0(O)o^A + \omega^1(O)\iota^A, \pi_{1'}(O)o_{A'} - \pi_{0'}(O)\iota_{A'}), \quad (6.1.18)$$

it is evident from (6.1.12), (6.1.16), and (6.1.17) that

$$Z^\alpha = Z^\alpha \delta_\alpha^\alpha, \qquad (6.1.19)$$

with

$$Z^\alpha = (\omega^0(O), \omega^1(O), \pi_{0'}(O), \pi_{1'}(O)), \quad \alpha = 0, 1, 2, 3. \qquad (6.1.20)$$

From this and (6.1.15) we have the following explicit equations:

$$Z^0 = \omega^0(O), \quad Z^1 = \omega^1(O), \quad Z^2 = \pi_{0'}(O) = Z_{0'}, \quad Z^3 = \pi_{1'}(O) = Z_{1'}.$$
$$(6.1.21)$$

So Z^0 and Z^1 can be consistently interpreted *either* as the 0, 1 components of Z^α *or* as the components of the spinor part Z^A at O. We shall make the convention that components of spinor parts of any twistor (unless otherwise stated) are always to be evaluated at the origin.

Dual twistors

Since $[{}_0^1]$-twistor space is effectively the direct sum of the spaces of spinors of type ω^A and $\pi_{A'}$ at O, the dual, $[{}_1^0]$-twistor space, must effectively be the direct sum of spaces of spinors of type λ_A, $\mu^{A'}$ at O. Typically, we may write

$$W_\alpha \overset{o}{\leftrightarrow} (\lambda_A(O), \mu^{A'}(O)) \qquad (6.1.22)$$

and the scalar product must then be defined as

$$W_\alpha Z^\alpha = \lambda_A(O)\omega^A(O) + \mu^{A'}(O)\pi_{A'}(O). \tag{6.1.23}$$

Analogously to (6.1.13) we wish to represent $\begin{bmatrix}0\\1\end{bmatrix}$-twistors by two spinor fields λ_A and $\mu^{A'}$, so that the dependence on the origin O is removed. We write

$$W_\alpha = (\lambda_A, \mu^{A'}), \tag{6.1.24}$$

(6.1.22) giving the values at O. We require (6.1.23) to hold not just at O, but at every point of $\mathbb{M}$:

$$\begin{aligned}\lambda_A \omega^A + \mu^{A'}\pi_{A'} = W_\alpha Z^\alpha &= \lambda_A(O)\omega^A(O) + \mu^{A'}(O)\pi_{A'}(O)\\ &= \overset{\circ}{\lambda}_A \overset{\circ}{\omega}^A + \overset{\circ}{\mu}^A \overset{\circ}{\pi}_{A'},\end{aligned} \tag{6.1.25}$$

where, as before, $\overset{\circ}{\lambda}_A$ and $\overset{\circ}{\mu}^{A'}$ are constant spinors whose values at O are $\lambda_A(O)$ and $\mu^{A'}(O)$, respectively. Substituting (6.1.10) into this equation yields

$$\lambda_A(\overset{\circ}{\omega}^A - ix^{AA'}\overset{\circ}{\pi}_{A'}) + \mu^{A'}\overset{\circ}{\pi}_{A'} = \overset{\circ}{\lambda}_A \overset{\circ}{\omega}^A + \overset{\circ}{\mu}^{A'}\overset{\circ}{\pi}_{A'}.$$

And since this relation must hold for arbitrary constant $\overset{\circ}{\omega}^A$, $\overset{\circ}{\pi}_{A'}$, the 'coefficients' of these spinors must be equal; this gives the following form for the fields λ_A, $\mu^{A'}$:

$$\begin{aligned}\lambda_A &= \overset{\circ}{\lambda}_A,\\ \mu^{A'} &= \overset{\circ}{\mu}^{A'} + ix^{AA'}\overset{\circ}{\lambda}_A.\end{aligned} \tag{6.1.26}$$

We can verify at once from (6.1.26) that the field $\mu^{A'}$ satisfies (and is, in fact, the general solution of) the conjugate twistor equation

$$\nabla_A^{(A'}\mu^{B')} = 0, \tag{6.1.27}$$

and that, analogously to (6.1.9), λ_A can be obtained from $\mu^{A'}$ by

$$\nabla_{AA'}\mu^{B'} = i\varepsilon_{A'}{}^{B'}\lambda_A. \tag{6.1.28}$$

Thus, the λ_A in (6.1.24) is redundant and W_α is fully determined by $\mu^{A'}$. In fact, we can, alternatively to (6.1.24), identify W_α with $\mu^{A'}$ (*cf.* (6.1.11)) and write

$$W_\alpha = [\mu^{A'}], \tag{6.1.29}$$

this being conformally invariant, like (6.1.11), though less convenient than (6.1.24) for building up twistors of higher valences.

It is worth noting the form which the inner product $W_\alpha Z^\alpha$ takes directly in terms of the spinor fields ω^A and μ^A. We need only substitute from (6.1.9) and (6.1.28) into (6.1.25) to find

$$[\mu^{A'}]\cdot[\omega^A] := W_\alpha Z^\alpha = \tfrac{1}{2}i\,(\mu^{A'}\nabla_{BA'}\omega^B - \omega^A\nabla_{AB'}\mu^{B'}). \tag{6.1.30}$$

Each solution $\mu^{A'}$ of (6.1.27) can be obtained by simply complex-

conjugating a solution of (6.1.1): $\omega^A \mapsto \mu^{A'} = \bar{\omega}^{A'}$. This is evident from a simple inspection of the two differential equations, or, alternatively, from an inspection of their respective general solutions (6.1.10), (6.1.26). This suggests that we identify the $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$-twistors W_α with the complex conjugates of the $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$-twistors Z^α, and vice versa. Consequently we define

$$\overline{Z^\alpha} = \overline{Z}_\alpha := (\bar{\pi}_A, \bar{\omega}^{A'}),$$
$$\overline{W_\alpha} = \overline{W}^\alpha := (\bar{\mu}^A, \bar{\lambda}_{A'}). \tag{6.1.31}$$

Note that the choice of the arbitrary factor in (6.1.9) as $-$ i enables us to pass via complex conjugation from (6.1.9) to (6.1.28), and from (6.1.10) to (6.1.26). The complex conjugation of an *arbitrary* twistor will be discussed presently.

Analogously to (6.1.15), we sometimes write

$$W_A = \lambda_A, \quad W^{A'} = \mu^{A'}, \tag{6.1.32}$$

(either as spinor fields or as point-spinors at O) for the spinor parts of W_α.

We shall want to define a dual basis δ^α_α to (6.1.17). It must satisfy

$$\delta^\alpha_\alpha \delta^\alpha_\beta = \delta^\alpha_\beta, \tag{6.1.33}$$

and this is easily verified for

$$\delta^0_\alpha \overset{O}{\leftrightarrow} (-\iota_A, 0) \qquad \delta^1_\alpha \overset{O}{\leftrightarrow} (o_A, 0)$$
$$\delta^2_\alpha \overset{O}{\leftrightarrow} (0, o^{A'}) \qquad \delta^3_\alpha \overset{O}{\leftrightarrow} (0, \iota^{A'}). \tag{6.1.34}$$

We then find that

$$W_\alpha = W_\alpha \delta^\alpha_\alpha, \tag{6.1.35}$$

with

$$W_\alpha = (\lambda_0(O), \lambda_1(O), \mu^{0'}(O), \mu^{1'}(O)), \quad \alpha = 0, 1, 2, 3. \tag{6.1.36}$$

Explicitly, from this and (6.1.32), we have

$$W_0 = \lambda_0(O), \quad W_1 = \lambda_1(O), \quad W_2 = \mu^{0'}(O) = W^{0'}, \quad W_3 = \mu^{1'}(O) = W^{1'}. \tag{6.1.37}$$

(Compare the remarks after (6.1.21).)

Higher-valence twistors

Now consider the outer product $X^\alpha Z^\beta$ of two twistors such as

$$X^\alpha = (\xi^A, \eta_{A'}), \quad Z^\alpha = (\omega^A, \pi_{A'}), \tag{6.1.38}$$

represented at O by

$$X^\alpha \overset{O}{\leftrightarrow} (\xi^A(O), \eta_{A'}(O)), \quad Z^\alpha \overset{O}{\leftrightarrow} (\omega^A(O), \pi_{A'}(O)). \tag{6.1.39}$$

Reference to (6.1.20) shows that its components $X^\alpha Z^\beta$ will consist of all the components at O of the following four spinors:

$$\xi^A \omega^B, \quad \xi^A \pi_{B'}, \quad \eta_{A'} \omega^B, \quad \eta_{A'} \pi_{B'}. \tag{6.1.40}$$

The spinor fields (6.1.40) are the spinor parts of $X^\alpha Z^\beta$ and they have a position dependence determined by (6.1.10) (as applied to (6.1.38)). The general $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$-twistor $S^{\alpha\beta}$, being a sum of products of type $X^\alpha Z^\beta$, will be fully characterized by four independent spinors at O, namely the values at O of the four fields

$$S^{AB}, \quad S^A{}_{B'}, \quad S_{A'}{}^B, \quad S_{A'B'}. \tag{6.1.41}$$

These are said to constitute the spinor parts of $S^{\alpha\beta}$. They are of a very special interrelated type, being sums of expressions as in (6.1.40), whose constituents (6.1.38) have the position dependence (6.1.10).

In using the notation (6.1.41) it is vital to keep the *order* of the spinor indices unchanged, e.g., never to write $S^A{}_{B'} = S_{B'}{}^A$ (contrary to our usual conventions, *cf.* (2.5.33), since this order is our only notational indication of which spinor part is meant. But the notation may become confusing when indices are raised and lowered with εs. For this reason, if the spinor parts of some twistor are to be used extensively it is often convenient to introduce separate symbols for the various spinor parts. But for the general discussion of twistors the present notation is very economical. We often write

$$S^{\alpha\beta} = \begin{pmatrix} S^{AB} & S^A{}_{B'} \\ S_{A'}{}^B & S_{A'B'} \end{pmatrix}. \tag{6.1.42}$$

In the same way we find the typical patterns* of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$- and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$-twistors:

$$E^\alpha{}_\beta = \begin{pmatrix} E^A{}_B & E^{AB'} \\ E_{A'B} & E_{A'}{}^{B'} \end{pmatrix}, \quad R_{\alpha\beta} = \begin{pmatrix} R_{AB} & R_A{}^{B'} \\ R^{A'}{}_B & R^{A'B'} \end{pmatrix}. \tag{6.1.43}$$

It is clear that the general $\begin{bmatrix} p \\ q \end{bmatrix}$-twistor will have 2^{p+q} spinor parts, which, however, cannot be exhibited as conveniently as the above. For example, a $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$-twistor $T^{\alpha\beta}{}_\gamma$ has eight independent spinor parts of the following form:

$$T^{AB}{}_C, \quad T_{A'}{}^B{}_C, \quad T^A{}_{B'C}, \quad T^{ABC'}, \quad T^A{}_{B'}{}^{C'}, \quad T_{A'}{}^{BC'}, \quad T_{A'B'C}, \quad T_{A'B'}{}^{C'}. \tag{6.1.44}$$

The particular spinor part which has all its indices at the upper level (in the

case above, $T^{ABC'}$) is called the *primary spinor part* of the twistor. That part with all lower indices is called the *projection part*.

The definition (6.1.23) of $W_\alpha Z^\alpha$ leads to definitions for contractions of arbitrary twistors. In practice this amounts to contracting the 'relevant' spinor parts over A and A' for each contracted twistor index α. For example,

$$T^{\alpha\beta}{}_\alpha = (T^{AB}{}_A + T_{A'}{}^{BA'}, \quad T^A{}_{B'A} + T_{A'B'}{}^{A'}). \tag{6.1.45}$$

It is now easy to see how we can take components of general twistors relative to the basis (6.1.17), analogously to (6.1.21) and (6.1.37). In the general case, exemplified by

$$T^{\alpha\beta}{}_\gamma = T^{\alpha\beta}{}_\gamma \delta^\alpha_\alpha \delta^\beta_\beta \delta^\gamma_\gamma, \tag{6.1.46}$$

we find (spinor field components being taken at O, in accordance with our remark after (6.1.21))

$$T^{00}{}_0 = T^{00}{}_0, \quad T^{01}{}_1 = T^{01}{}_1, \text{ etc.,}$$
$$T^{20}{}_1 = T_{0'}{}^0{}_1, \quad T^{20}{}_3 = T_{0'}{}^{01'}, \quad T^{31}{}_2 = T_1{}^{,10'}, \text{ etc.,} \tag{6.1.47}$$

where the *left* member of each equation is the *twistor* component, and the *right* member is a spinor-part component. Since twistor indices are not primed and spinor indices are never 2 or 3, there is no ambiguity of meaning in the second line of (6.1.47). In the first line, where there might be ambiguity, there is in fact none. Indeed, we have the following rule:

$$0(\text{up/down}) \leftrightarrow 0(\text{up/down}), \quad 1(\text{up/down}) \leftrightarrow 1(\text{up/down})$$
$$2(\text{up/down}) \leftrightarrow 0'(\text{down/up}), \quad 3(\text{up/down}) \leftrightarrow 1'(\text{down/up}) \tag{6.1.48}$$

We next examine the position dependence of the spinor parts (6.1.41) of the $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$-twistor $S^{\alpha\beta}$. The mutual and position dependences of the (field-)spinor parts of 1-valent twistors determine the form of the (field-)spinor parts of all twistors. For $S^{\alpha\beta}$, these can be found from the requirement that, for two arbitrary $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$-twistors U_α, W_α, the (scalar) field represented by

$$S^{\alpha\beta} U_\alpha W_\beta \tag{6.1.49}$$

is constant and is therefore equal to its value at the origin. Proceeding exactly as in (6.1.25), we easily find the desired relations. In order to exhibit these conveniently we prefer to introduce a more specific notation for the field-spinor parts of $S^{\alpha\beta}$, namely

$$S^{\alpha\beta} = \begin{pmatrix} \sigma^{AB} & \rho^A{}_{B'} \\ \tau_{A'}{}^B & \kappa_{A'B'} \end{pmatrix}, \tag{6.1.50}$$

(although S^{AB}, $S^A{}_{B'}$, etc. would be perfectly legitimate). Then the relations

between these are found to be as follows:

$$\kappa_{A'B'} = \overset{\circ}{\kappa}_{A'B'},$$
$$\tau^B_{A'} = \overset{\circ}{\tau}^B_{A'} - \mathrm{i}x^{BB'}\overset{\circ}{\kappa}_{A'B'},$$
$$\rho^A_{B'} = \overset{\circ}{\rho}^A_{B'} - \mathrm{i}x^{AA'}\overset{\circ}{\kappa}_{A'B'},$$
$$\sigma^{AB} = \overset{\circ}{\sigma}^{AB} - \mathrm{i}x^{AA'}\overset{\circ}{\tau}^B_{A'} - \mathrm{i}x^{BB'}\overset{\circ}{\rho}^A_{B'} - x^{AA'}x^{BB'}\overset{\circ}{\kappa}_{A'B'}.$$

(6.1.51)

The content of (6.1.51) can also be expressed in terms of the following differential equations (*cf.* (6.1.9), (6.1.28)):

$$\nabla_{CC'}\sigma^{AB} = -\mathrm{i}\varepsilon_C{}^A\tau^B_{C'} - \mathrm{i}\varepsilon_C{}^B\rho^A_{C'},$$
$$\nabla_{CC'}\rho^A_{B'} = -\mathrm{i}\varepsilon_C{}^A\kappa_{C'B'},$$
$$\nabla_{CC'}\tau^B_{A'} = -\mathrm{i}\varepsilon_C{}^B\kappa_{A'C'},$$
$$\nabla_{CC'}\kappa_{A'B'} = 0.$$

(6.1.52)

One way to establish these is to differentiate (6.1.51) at the origin, and then to recall that the origin is arbitrary, so that (6.1.52) holds generally. In this special case the twistor $S^{\alpha\beta}$ is still fully determined by a single field, namely by its primary spinor part σ^{AB}. For (6.1.52)(1) yields

$$\nabla_{CC'}\sigma^{CB} = -2\mathrm{i}\tau^B_{C'} - \mathrm{i}\rho^B_{C'},$$
$$\nabla_{CC'}\sigma^{BC} = -\mathrm{i}\tau^B_{C'} - 2\mathrm{i}\rho^B_{C'},$$

(6.1.53)

whence we get $\tau^B_{C'}$ and $\rho^B_{C'}$. And these, via (6.1.52)(2) or (3), yield $\kappa_{A'B'}$.

Relations analogous to (6.1.51) and (6.1.52) evidently hold for twistors of *any* valence, and can be obtained in an analogous way. Let us consider one more special case, namely the $\begin{bmatrix}1\\1\end{bmatrix}$-twistor $E^\alpha{}_\beta$ of (6.1.43) for whose field-spinor parts we now again introduce a more specific notation:

$$E^\alpha{}_\beta = \begin{pmatrix} \theta^A{}_B & \xi^{AB'} \\ \eta_{A'B} & \zeta_{A'}{}^{B'} \end{pmatrix}.$$

(6.1.54)

For these fields we find

$$\eta_{A'B} = \overset{\circ}{\eta}_{A'B},$$
$$\zeta_{A'}{}^{B'} = \overset{\circ}{\zeta}_{A'}{}^{B'} + \mathrm{i}x^{BB'}\overset{\circ}{\eta}_{A'B},$$
$$\theta^A{}_B = \overset{\circ}{\theta}^A{}_B - \mathrm{i}x^{AA'}\overset{\circ}{\eta}_{A'B},$$
$$\xi^{AB'} = \overset{\circ}{\xi}^{AB'} - \mathrm{i}x^{AA'}\overset{\circ}{\zeta}_{A'}{}^{B'} + \mathrm{i}x^{BB'}\overset{\circ}{\theta}^A{}_B + x^{AA'}x^{BB'}\overset{\circ}{\eta}_{A'B},$$

(6.1.55)

which (analogously to (6.1.51), (6.1.52)) are equivalent to

$$\nabla_{CC'}\xi^{AB'} = \mathrm{i}\varepsilon_C{}^{B'}\theta^A{}_C - \mathrm{i}\varepsilon_C{}^A\zeta_C{}^{B'},$$
$$\nabla_{CC'}\theta^A{}_B = -\mathrm{i}\varepsilon_C{}^A\eta_{C'B},$$
$$\nabla_{CC'}\zeta_{A'}{}^{B'} = \mathrm{i}\varepsilon_C{}^{B'}\eta_{A'C},$$
$$\nabla_{CC'}\eta_{A'B} = 0.$$

(6.1.56)

We note that equations (6.1.55) remain valid if we subject $\theta^A{}_C$ and $\zeta_C{}^{B'}$ to the changes

$$\theta^A{}_B \mapsto \theta^A{}_C + \lambda\varepsilon_C{}^A,$$
$$\zeta_C{}^{B'} \mapsto \zeta_C{}^{B'} + \lambda\varepsilon_C{}^{B'} \quad (\lambda = \text{constant}), \tag{6.1.57}$$

and so we see that in this case the primary spinor part $\xi^{AB'}$ of $\mathsf{E}^\alpha{}_\beta$ does *not* uniquely determine $\mathsf{E}^\alpha{}_\beta$. The transformation (6.1.57) in fact changes the trace

$$\mathsf{E}^\alpha{}_\alpha = \theta^A{}_A + \zeta_{A'}{}^{A'} \tag{6.1.58}$$

of the twistor:

$$\mathsf{E}^\alpha{}_\alpha \mapsto \mathsf{E}^\alpha{}_\alpha + 4\lambda. \tag{6.1.59}$$

Only if the trace is independently known (e.g., known to be zero), is $\mathsf{E}^\alpha{}_\beta$ in fact determined by $\xi^{AB'}$.

Equations analogous to (6.1.51), (6.1.55) hold for all twistors. For some, such as for $\mathsf{S}^{\alpha\beta}$, all the information of the twistor is contained in the primary part. But the example of $\mathsf{E}^\alpha{}_\beta$ shows that this need not be the case. A class of twistors for which the primary part *does* carry all the information is that of the trace-free symmetric twistors:

$$\mathsf{T}^{\alpha\ldots\delta}{}_{\rho\ldots\tau} = \mathsf{T}^{(\alpha\ldots\delta)}{}_{(\rho\ldots\tau)}, \tag{6.1.60}$$

$$\mathsf{T}^{\alpha\beta\ldots\delta}{}_{\alpha\sigma\ldots\tau} = 0. \tag{6.1.61}$$

The equation satisfied by the primary part $\mathsf{T}^{A\ldots DR'\ldots T'} = \lambda^{A\ldots DR'\ldots T'}$ is

$$\nabla^{(E}_{(U'}\lambda^{A\ldots D)}_{R'\ldots T')} = 0. \tag{6.1.62}$$

At the other extreme, the alternating twistors $\varepsilon_{\alpha\beta\gamma\delta}$, $\varepsilon^{\alpha\beta\gamma\delta}$, satisfying

$$\varepsilon_{\alpha\beta\gamma\delta} = \varepsilon_{[\alpha\beta\gamma\delta]}, \quad \varepsilon^{\alpha\beta\gamma\delta} = \varepsilon^{[\alpha\beta\gamma\delta]}, \quad \varepsilon_{0123} = \varepsilon^{0123} = 1, \tag{6.1.63}$$

each have only six non-zero spinor parts, out of a total of sixteen, namely the parts with two unprimed and two primed indices. These are

$$\varepsilon^{A'B'}{}_{CD} = \varepsilon^{A'B'}\varepsilon_{CD}, \quad \varepsilon^{A'}{}_B{}^{C'}{}_D = -\varepsilon^{A'C'}\varepsilon_{BD}, \text{ etc.,} \tag{6.1.64}$$

for $\varepsilon_{\alpha\beta\gamma\delta}$, and

$$\varepsilon_{A'B'}{}^{CD} = \varepsilon_{A'B'}\varepsilon^{CD}, \quad \varepsilon_{A'}{}^B{}_{C'}{}^D = -\varepsilon_{A'C'}\varepsilon^{BD}, \text{ etc.,} \tag{6.1.65}$$

for $\varepsilon^{\alpha\beta\gamma\delta}$. In these cases the primary part *vanishes*. (It also vanishes in the case of a skew twistor $\mathsf{X}^{\alpha\beta\gamma} = \mathsf{X}^{[\alpha\beta\gamma]}$.) In fact, *only* the trace-free symmetric twistors can be represented in this way by a *single* spinor field (namely by their primary spinor part) subject to a single first-order differential equation. In certain other cases (e.g., that of a skew $[{}^2_0]$-twistor) this primary spinor part does determine the twistor completely, but is not characterized by a first-order differential equation. In most other cases the primary spinor

part is insufficient to define the twistor. A symmetric twistor (6.1.60) which is *not* trace-free also has a primary part which satisfies (6.1.62), but the various trace parts, e.g., expressions of the form $\delta^{(\alpha}_{(\rho} U^{\beta\cdots\delta)}_{\sigma\cdots\tau)}$, are not determined by the primary part (i.e., such expressions can be added to $T^{\alpha\cdots\delta}{}_{\rho\cdots\tau}$ without changing the primary part).

We shall be particularly interested in *symmetric, skew-symmetric,* and *Hermitian* 2-valent twistors. The twistor $S^{\alpha\beta}$ of (6.1.50) is symmetric if $S^{\alpha\beta} = S^{\beta\alpha}$, i.e., if

$$\sigma^{AB} = \sigma^{BA}, \quad \rho^A_{B'} = \tau^A_{B'}, \quad \kappa_{A'B'} = \kappa_{B'A'}. \tag{6.1.66}$$

It can be seen from (6.1.52) that the mere symmetry of σ^{AB} at all points forces the second and third of equations (6.1.66), and is therefore sufficient for the symmetry of $S^{\alpha\beta}$. For a skew-symmetric twistor ($S^{\alpha\beta} = -S^{\beta\alpha}$) we have a minus sign on the right of all equations (6.1.66), and again the mere skew-symmetry of σ^{AB} at *all* points forces the rest.

We say that the twistor $E^\alpha{}_\beta$ of (6.1.54) is *Hermitian* if

$$\bar{E}_\beta{}^\alpha = E^\alpha{}_\beta, \tag{6.1.67}$$

i.e., if

$$\xi^{AB'} = \bar{\xi}^{AB'}, \quad \eta_{AB'} = \bar{\eta}_{AB'}, \quad \theta^A{}_B = \bar{\zeta}_B{}^A. \tag{6.1.68}$$

Analogously to the remark we made about symmetric spinors, it can now be seen from (6.1.55) that the Hermiticity of $\xi^{AB'}$ at *all points* forces the second of equations (6.1.68), but permits the members of (6.1.68)(3) to differ by an imaginary multiple of $\varepsilon_B{}^A$. Consequently, however, it suffices to ensure the Hermiticity of $E^\alpha{}_\beta$ if $E^\alpha{}_\beta$ is known to be trace-free.

An important observation is the following: the primary spinor part σ^{AB} of *any* twistor $S^{\alpha\beta}$ automatically satisfies the differential equation

$$\nabla^{(C}_{C'} \sigma^{AB)} = 0, \tag{6.1.69}$$

as is clear from (6.1.52)(1). Also the primary spinor part $\xi^{AB'}$ of any twistor $E^\alpha{}_\beta$ automatically satisfies the differential equation

$$\nabla^{(B}_{(B'} \xi^{A)}_{A')} = 0 \tag{6.1.70}$$

(which is actually the conformal Killing equation, see §6.5 below), as follows from (6.1.56)(1). Just as with (6.1.1), which is satisfied by the primary spinor part of Z^α, these two differential equations are conformally invariant, with

$$\hat{\sigma}^{AB} = \sigma^{AB}, \quad \hat{\xi}^{AB'} = \xi^{AB'} \tag{6.1.71}$$

(as is implicit, in the case of conformally flat space, in their twistor origins). This can be established independently of space–time flatness as for (6.1.1). Furthermore, by an argument analogous to that leading to (6.1.10), it can be shown that the general solutions in $\mathbb{M}$ of equations (6.1.69) and (6.1.70) are,

with σ^{AB} symmetric, given by (6.1.51) (4), (6.1.55) (4), respectively. Note that (6.1.69) and (6.1.70) are both special cases of (6.1.62), which is again conformally invariant (*cf.* (5.6.15)), with

$$\hat{\chi}^{A...CR'...T'} = \chi^{A...CR'...T'}. \tag{6.1.72}$$

Equations (6.1.69), (6.1.70) and (6.1.62) will be further discussed below (see §6.7; also (6.4.1)). Arguments from §6.7 show that *all symmetric solutions of* (6.1.62) *are primary parts of trace-free symmetric twistors.*

We now return to the question of defining complex conjugation of the general twistor. The rule is, in fact, determined by the definitions (6.1.31) for 1-valent twistors, together with the requirement that complex conjugation commute with product and sum, e.g.,

$$\overline{V^\alpha W_\beta + X^\alpha Y_\beta} = \overline{V^\alpha W_\beta} + \overline{X^\alpha Y_\beta}$$

Consider, for example, the twistor

$$P^\alpha{}_\beta = Z^\alpha W_\beta = \begin{pmatrix} Z^A W_B & Z^A W^{B'} \\ Z_{A'} W_B & Z_{A'} W^{B'} \end{pmatrix}.$$

We must have

$$\overline{P^\alpha{}_\beta} = \overline{P}_\alpha{}^\beta = \overline{Z}_\alpha \overline{W}^\beta = \begin{pmatrix} \overline{Z}_A \overline{W}^B & \overline{Z}_A \overline{W}_{B'} \\ \overline{Z}^{A'} \overline{W}^B & \overline{Z}^{A'} \overline{W}_{B'} \end{pmatrix}. \tag{6.1.73}$$

Since the general $[{}^1_1]$-twistor is a sum of twistors of the type of $P^\alpha{}_\beta$, and since the most general $[{}^p_q]$-twistor can be dealt with analogously, we recognize the following general rule: *To conjugate a twistor, we conjugate all its spinor parts, and then place each conjugated part into the correct position, namely that appropriate for a twistor with all original twistor indices at reverse level.*

Conformal invariance of helicity and scalar product

We define the *helicity* of a twistor Z^α by

$$s := \tfrac{1}{2} Z^\alpha \overline{Z}_\alpha = \tfrac{1}{2}(\omega^A \bar\pi_A + \pi_{A'} \bar\omega^{A'}) \tag{6.1.74}$$

(likewise, for W_α it is $s = \tfrac{1}{2}\overline{W}^\alpha W_\alpha$), and this is evidently real, although it can be positive or negative. We say Z^α (or W_α) is *null* if its helicity vanishes, *right handed* if $s > 0$, and *left handed* if $s < 0$. Twistor space $\mathbb{T}^\bullet(= \mathbb{T} = \mathbb{T}^\alpha)$ is thus composed of three pieces $\mathbb{T}^0 = \mathbb{N}$, $\mathbb{T}^+$ and $\mathbb{T}^-$, consisting of null, right-handed and left-handed twistors, respectively. Similarly, dual twistor space $\mathbb{T}_\bullet(= \mathbb{T}^* = \mathbb{T}_\alpha)$ is composed of $\mathbb{T}_0$, $\mathbb{T}_+$ and $\mathbb{T}_-$. For the *projective* versions of these spaces, i.e. the systems of one-dimensional linear subspaces contained in them (together with the origin), the prefix $\mathbb{P}$ is adjoined (see Fig. 6-1 and §9.3).
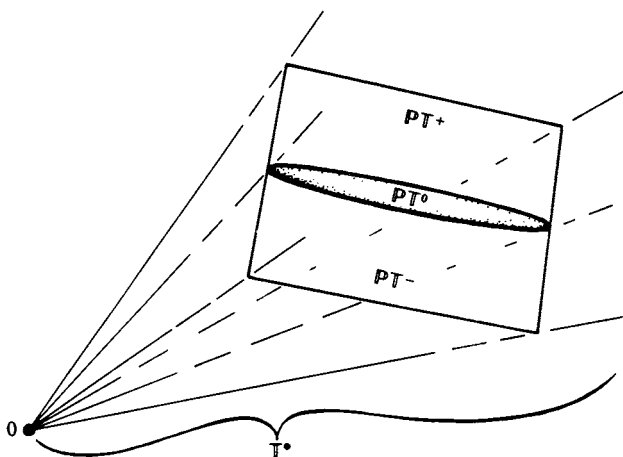
Fig. 6-1. Projective twistor space $\mathbb{PT}^\bullet$ is the space of one-dimensional linear subspaces of twistor space $\mathbb{T}^\bullet$. It consists of three regions $\mathbb{PT}^+$, $\mathbb{PT}^-$ and $\mathbb{PT}^0\ (=\mathbb{PN}^\bullet)$.

From (6.1.9) and (6.1.3) we see that, under conformal rescaling, the second of the following equations holds; the first is simply (6.1.2) again:

$$\hat{\omega}^A = \omega^A$$
$$\hat{\pi}_{A'} = \pi_{A'} + i\Upsilon_{AA'}\omega^A. \tag{6.1.75}$$

Thus $\pi_{A'}$ is *not* a conformally weighted spinor field. (Note the formal similarity of the above equation with (6.1.10).) Equation (6.1.75) describes the effects of changes of the conformal scaling in the given manifold $\mathbb{M}$. The twistor Z^α itself is regarded as unaffected by this change. But its representation by spinor parts changes, unless we adopt an appropriate view of the spinor $\pi_{A'}$. In fact, $\pi_{A'}$ can be regarded not as a conformally weighted spinor field, but as something that behaves in the more complicated way (6.1.75) under conformal rescaling of the metric. (Viewed in this way, $\pi_{A'}$ cannot be considered independently of ω^A.) With this interpretation, it is still legitimate to write $Z^\alpha = (\hat{\omega}^A, \hat{\pi}_{A'})$. This viewpoint is the one we shall adopt when we consider local twistors in §6.9.

The analogue of (6.1.75) for the spinor parts of a $[^0_1]$-twistor W_α is

$$\hat{\lambda}_A = \lambda_A - i\Upsilon_{AA'}\mu^{A'},$$
$$\hat{\mu}^{A'} = \mu^{A'}, \tag{6.1.76}$$

(*cf.* (6.1.26)). We can now immediately verify the conformal invariance of the twistor inner product (6.1.23) and hence of the helicity given by (6.1.74):

$$\hat{\lambda}_A \hat{\omega}^A + \hat{\mu}^{A'} \hat{\pi}_{A'} = (\lambda_A - i\Upsilon_{AA'}\mu^{A'})\omega^A + \mu^{A'}(\pi_{A'} + i\Upsilon_{AA'}\omega^A)$$
$$= \lambda_A \omega^A + \mu^{A'} \pi_{A'} = \mathsf{W}_\alpha \mathsf{Z}^\alpha. \tag{6.1.77}$$

Thus the twistor inner product is purely a property of twistor space and is independent of any particular point in space–time or choice of conformal scale.

6.2 Some geometrical aspects of twistor algebra

The geometrical meaning of twistors is clearest in the case of *null* $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$-twistors:

$$\mathsf{Z}^\alpha \bar{\mathsf{Z}}_\alpha = 0. \tag{6.2.1}$$

Suppose we have a particular null twistor $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$ with $\pi_{A'} \neq 0$. Let us first determine the locus Z of points in $\mathbb{M}$ at which $\omega^A = 0$: the geometry of the field ω^A is best described in terms of this locus. On Z the position vector must satisfy (*cf.* (6.1.10))

$$ix^{AA'} \mathring{\pi}_{A'} = \mathring{\omega}^A. \tag{6.2.2}$$

We shall wish to assume that ω^A and $\bar{\pi}^A$ are not proportional at O. If by chance they *are*, we use the freedom we had in solving (6.1.9) for (6.1.10), and choose a *different* point as origin, where ω^A and $\bar{\pi}^A$ are *not* proportional. (This is always possible, since, by (6.1.10), $\bar{\pi}_A \omega^A = \bar{\pi}_A \mathring{\omega}^A - ix^{AA'} \bar{\pi}_A \pi_{A'}$; so if $\bar{\pi}_A \mathring{\omega}^A = 0$ we need merely go to a point at which $x^{AA'} \bar{\pi}_A \pi_{A'} \neq 0$ to achieve $\bar{\pi}_A \omega^A \neq 0$.) Assume this done. Then a particular solution of (6.2.2) is given by

$$x^a = (i\mathring{\omega}^{B'} \pi_{B'})^{-1} \mathring{\omega}^A \mathring{\omega}^{A'}. \tag{6.2.3}$$

This vector is real, since the parenthesis is real, by (6.2.1) and (6.1.74). The remaining solutions of (6.2.2) must be such that their differences from (6.2.3) annihilate $\pi_{A'}$. So since x^a is real, these differences must be real multiples of $\bar{\pi}^A \pi^{A'}$. Consequently the general solution of (6.2.2) has the form

$$x^a = (i\mathring{\omega}^{B'} \pi_{B'})^{-1} \mathring{\omega}^A \mathring{\omega}^{A'} + h\bar{\pi}^A \pi^{A'}, \quad h \in \mathbb{R}. \tag{6.2.4}$$

This describes a *null straight line* Z, hereafter called a *ray*, in the direction of the flagpole of $\bar{\pi}^A$; it passes through a point Q, given by $h = 0$ in (6.2.4) whose displacement from O is along the flagpole of $\mathring{\omega}^A$ so Q lies on the light cone of O (see Fig. 6-2).

Note that the ray Z is independent of the scaling of Z^α: if we replace Z^α by $\lambda \mathsf{Z}^\alpha$ ($\lambda \neq 0$) then Z is unchanged. Conversely we easily see that the ray Z *determines* Z^α up to proportionality, since (6.2.2) is homogeneous in $\mathring{\omega}^A$, $\mathring{\pi}_{A'}$.
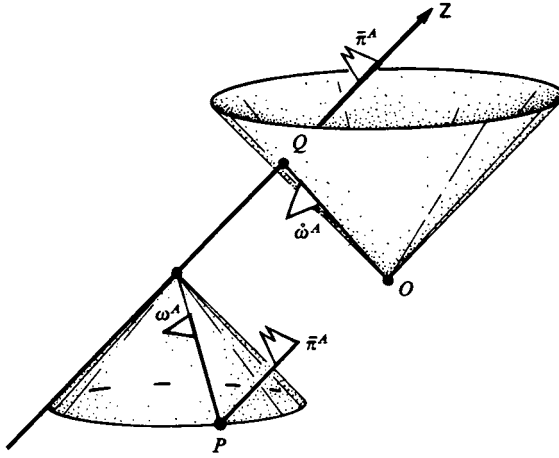
Fig. 6-2. The ray Z, determined by a null twistor Z^α, points in the direction of the flagpole of $\bar\pi^A$ and passes through a point Q whose displacement from the origin O is in the direction of the flagpole of $\mathring{\omega}^A$. At a general point P, the flagpole of ω^A lies in a ray meeting Z.

If $\pi_{A'} = 0$ there is *no* finite locus Z (*cf.* (6.2.2)). Instead (provided $\mathring{\omega}^A$ does not also vanish, in which case $\omega^A \equiv 0$ and so $Z^\alpha = 0$) one can then interpret the locus Z as a generator of the 'light cone at infinity'. This will be discussed in Chapter 9.

Flagpole field of ω^A: Robinson congruence

Having found the locus Z, it is easy to describe the general geometric pattern of the flagpoles of the field ω^A. Recalling the freedom we have in the choice of origin O, we see that a construction similar to the above can be carried out at any point P at which ω^A is not proportional to $\bar\pi^A$. Where it *is* proportional, the flagpole of ω^A is in a direction parallel to Z. For a general point P however, the flagpole direction of ω^A will lie along the (unique) ray joining P to the point at which Z cuts its light cone. Thus the field of flagpole directions of ω^A consists simply of all the null directions in all the light cones whose vertices lie on Z, together with those in the limiting light cone where the vertex goes to infinity on Z – the (unique) null hyperplane containing Z. This hyperplane is the locus of points at which the flagpole direction of ω^A is parallel to Z (ω^A proportional to $\bar\pi^A$).

If $Z^\alpha = (\omega^A, \pi_{A'})$ is not null, it is still possible to regard it as representing a locus in 'complexified' space–time. This viewpoint will be pursued in §9.3.

However, a realization of Z^α in real terms can also be given. It turns out that the flagpole directions of ω^A still lie in a congruence of real rays. The rays twist about one another (without shear) in a right-handed or left-handed sense according as $Z^\alpha \bar{Z}_\alpha$ is positive or negative. (The shear-free – and geodetic, i.e., straight-line – property of the congruence is a consequence of the equation $\omega^A \omega^B \nabla_{AA'} \omega_B = 0$ which follows from (6.1.9); see §7.1.) Congruences arising from twistors in this way are called *Robinson congruences* (*cf* Penrose 1967*a*). Knowledge of the Robinson congruence associated with a twistor fixes the twistor up to a scalar multiplier.

There is another way in which the Robinson congruence associated with a twistor Z arises. Consider the particular ray X through O in the flagpole direction of $\omega^A(O)$. This can be represented by a twistor

$$X^\alpha \overset{O}{\leftrightarrow} (0, \bar{\omega}_{A'}(O)),$$

or by any non-zero multiple of this. Evidently $X^\alpha \bar{Z}_\alpha = 0$, which condition characterizes, at O, the flagpole direction of ω^A. But the origin is arbitrary, so at *any* point the flagpole direction of ω^A is the direction of a ray X through that point described by a twistor X^α subject to

$$X^\alpha \bar{Z}_\alpha = 0 = X^\alpha \bar{X}_\alpha, \qquad (6.2.5)$$

X^α being, of course, necessarily null. Thus the flagpoles of the ω-field all point along the rays given by (6.2.5) (for fixed $\bar{Z}_\alpha$), whence (6.2.5) describes the Robinson congruence.

To get a picture of a Robinson congruence we choose a particular $Z^\alpha = (\omega^A, \pi_{A'})$ with $Z^\alpha \bar{Z}_\alpha = 2s$, given in the standard coordinate system and spin-frame, *cf.* (3.1.31) (and Chapter 1), by

$$Z^\alpha = (0, s, 0, 1) \quad (s \in \mathbb{R}). \qquad (6.2.6)$$

The equation of the ω-field is given (*cf.* (6.1.10)) by

$$(\omega^0, \omega^1) = (0, s) - \frac{i}{\sqrt{2}} \begin{pmatrix} t + z & x + iy \\ x - iy & t - z \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad (6.2.7)$$

so

$$\omega^0 : \omega^1 = x + iy : t - z + is\sqrt{2}. \qquad (6.2.8)$$

We can find a differential equation for the rays of the congruence, the tangent direction defined by $dt : dx : dy : dz$ being that of the flagpole of ω^A, i.e.

$$dt + dz : dx + idy = dx - idy : dt - dz$$
$$= \bar{\omega}^{0'} : \bar{\omega}^{1'} \qquad (6.2.9)$$
$$= x - iy : t - z - is\sqrt{2}.$$

The general solution of these equations can be written down directly. But it